Twisted string bordism in 7 dimensions with applications to anomaly cancellation

Natalia María Pacheco-Tallaj
nataliap@mit.edu

based on joint work with I. Basile, C. Krulewski, G. Leone

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Outline

Anomalies in quantum field theory

What is a field theory?

A field theory is the data of

- A *spacetime manifold* X which is d dimensional
- A moduli $\mathcal{F}(X)$ of objects over X, called *fields*, e.g.
  - connections on a principal G-bundle $P \to X$ (gauge fields)
  - sections of a vector bundle $E \to X$ (matter fields)
  - maps to a target manifold $\mathrm{Maps}(X, Y)$ (sigma models)
  - metrics on X (gravity)
- An *action functional* $S : \mathcal{F}(X) \to \mathbb{C}$

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A classical field theory finds physical fields ψ by solving a principle of stationary action $\delta S/\delta\psi = 0$.

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A quantum field theory instead defines a probability measure on $\mathcal{F}(X)$ weighted by $\exp(-S)$ (assuming X has Euclidean signature) and computes probability amplitudes and correlation functions of observable physical quantities in this measure.

Partition functions and path integrals

- QFT: probability distribution over space of fields weighted by $\exp(-S)$
- Path integral/partition function: integrate over dynamical fields ψ and consider result as a function of the background fields A

$$Z_X[A] = \int_{\psi \in \mathcal{F}^{\mathrm{dyn}}(X)} D\psi\, e^{-S[\psi, A]} : \mathcal{F}^{\mathrm{bg}}(X) \to \mathbb{C}$$

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(schematically $\mathcal{F}^{\mathrm{dyn}}(X) \to \mathcal{F}(X) \to \mathcal{F}^{\mathrm{bg}}(X)$)

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- Geometrically: $Z[A]$ is not a function over $\mathcal{F}^{\text{bg}}(X)$, but a section of an *anomaly line bundle* $\mathcal{L}_{\text{anom}}$
  - curvature = "local anomaly"
  - holonomy = "global anomaly"

What is a quantum anomaly?

"A failure of $Z[A]$ to be defined globally and gauge invariantly."

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- By the family index theorem, we have a class $P_{d+2} := [\hat{A}(X)\mathrm{ch}(F)]_{(d+2)}$ on $X \times [\mathcal{A}/\mathcal{G}]$ such that $\int_X P_{d+2} \in \Omega^2(\mathcal{A}/\mathcal{G})$ is the curvature form of $\mathcal{L}_{\text{anom}}$.

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Anomaly inflow

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$$Z_X[A]e^{-2\pi i \alpha_{\mathrm{anom}}}$$

- (Anomaly inflow) This phase comes from the partition function of a field theory in dimension $d+1$: there is some M with $\partial M = X$ and such that

$$Z_X[A]e^{-2\pi i(\mathrm{Idx}(M) + \int_M H \wedge X_8)} \text{ is gauge invariant}$$

  only well defined if X_4 is cohomologically trivial on M.
- We call this $d+1$-dimensional theory the anomaly theory of Z

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- Upshot: all the information of the anomalies is encoded in a $7d$ anomaly theory defined on 7-manifolds with $\frac{1}{2}p_1(TM) = nc_1(\mathcal{L})^2$

String structures

7-manifolds with $\frac{1}{2}p_1(TM) = nc_1(\mathcal{L})^2$

Definition (String structure)

Let M be a spin manifold, then $p_1(M)$ is even. A *string structure* on M is a trivialization of $\frac{1}{2}p_1(M)$.

$$\begin{tikzcd} & B\mathrm{String} \arrow[d] & \\ M \arrow[r, "TM"] \arrow[ur, dashed] & B\mathrm{Spin} \arrow[r, "\frac{1}{2}p_1"] & K(\mathbb{Z},4) \end{tikzcd}$$

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$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "TM"'] & BU(1) \arrow[d, "nc_1(\mathcal{O}(1))^2"] \\ B\mathrm{Spin} \arrow[r, "\frac{1}{2}p_1"] & K(\mathbb{Z},4) \end{tikzcd}$$

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Remark

Consider the (virtual) bundle $T = \mathcal{O}(1) + \mathcal{O}(-1) - 2_{\mathbb{C}}$ on $BU(1)$. We also call this a $(BU(1), -nT)$-twisted string structure, because given a manifold M with a map $f : M \to BU(1)$, it is the data of a string structure on $TM + f^*(-nT)$.[a]

[a] $\frac{1}{2}p_1(TM - f^*nT) = \frac{1}{2}p_1(TM) - nc_1(f^*\mathcal{O}(1))^2 = 0$

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- α_M is a well-defined field theory in $d+1$ dimensions if M is nc_1^2-twisted string
- We call it the *anomaly theory*, and it is an invertible topological field theory defined on nc_1^2-twisted string manifolds

Invertible field theories and bordism

The functorial formalism and its interpretation

The data of a field theory in $d+1$ dimensions can often be organized mathematically as follows:

Definition (Functorial field theory)

A $d+1$-dimensional functorial field theory (on nc_1^2-twisted String manifolds) is a symmetric monoidal functor

$$Z : (\mathrm{Bord}_{\langle d,d+1\rangle}^{nc_1^2\text{-String}}, \sqcup) \to (s\mathrm{Vect}_{\mathbb{C}}, \otimes)$$

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- $Z(X^d)$ is interpreted as a state space of the theory
- For a closed $d+1$ manifold M, interpreted as a bordism $\emptyset \rightsquigarrow \emptyset$, the morphism $Z(M) : \mathbb{C} \to \mathbb{C}$ in $s\mathrm{Vect}_{\mathbb{C}}$ is determined by a choice of complex number $\eta \in \mathbb{C}$ interpreted as the value of the partition function on M

Invertibility

There is a tensor product structure on the set of all functorial field theories given by point-wise tensor product in $(s\mathrm{Vect}_{\mathbb{C}}, \otimes)$:

$$(Z_1 \otimes Z_2)(X^d) = Z_1(X) \otimes Z_2(X)$$

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Definition (Invertible field theory)

A functorial field theory $Z : \mathrm{Bord}_{\langle d,d+1 \rangle}^{nc_1^2\text{-String}} \to s\mathrm{Vect}_{\mathbb{C}}$ is said to be *invertible* if it is tensor-invertible under the above tensor product.

Invertible field theories

Invertible field theories (IFTs) factor through the maximal Picard groupoid [1] of the target $s\mathrm{Line}_{\mathbb{C}} \subset s\mathrm{Vect}_{\mathbb{C}}$

$$
\begin{tikzcd}
(\mathrm{Bord}^{nc_1^2\text{-twisted}}_{\langle d,d+1\rangle}, \sqcup) \arrow[r, "Z"] \arrow[dr, dashed] & (s\mathrm{Vect}_{\mathbb{C}}, \otimes) \\
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[1] A Picard groupoid is a fully invertible symmetric monoidal category

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Invertible field theories (IFTs) factor through the groupoid completion of the source.

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Classifying invertible field theories

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Classification theorems: Freed-Hopkins, Freed-Hopkins-Teleman, Grady (deformation classes)

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  2. Construct 7-dimensional manifold generators to compute α_{anom} on

Twists of $BU(1)$-string bordism in dimension 7

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- Generators?
  - $K3 \times S^3$, $\mathbb{CP}^2 \times S^3$ are not twisted string,
  - odd dimension means no hope for complex algebraic representatives.

Challenges computing $\Omega_7^{\mathrm{String}-nc_1^2}$

- Adams spectral sequence: trivial at $p \geq 5$, extension problems at $p = 2, 3$.
- Few useful comparison maps, $\Omega_7^{\mathrm{Spin}} \cong \Omega_7^{\mathrm{String}} \cong 0$ and comparison maps changing the dimension land in high dimensional or 0 groups. Smith homomorphisms provide limited success for some $-nT$ twists but we want something that works for all $-nT$ twists.
- Generators?
    - $K3 \times S^3$, $\mathbb{CP}^2 \times S^3$ are not twisted string,
    - odd dimension means no hope for complex algebraic representatives.
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Solutions

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Solutions: characterize homotopically inequivalent twists

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Theorem (Basile–Krulewski–Leone–P.-T.)

The homotopy class of $M\mathrm{String} \wedge BU(1)^{-nT}$ only depends on the value of $n \pmod{12}$.

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Solutions: bordism invariants

Theorem (Basile–Krulewski–Leone–P.-T.)

Given a $(BU(1), -nT)$-twisted string 7-manifold M, there exists a spin 8-manifold N with $\widetilde{f} : N \to BU(1)$ and $\partial N = M$, then the map

$$\alpha_8^E : \Omega_8^{Spin}(BU(1)) \to \mathbb{Z}$$
$$N \mapsto \int_N \hat{A}(TN)ch(E - \mathrm{rk}\, E)$$

descends to an invariant

$$\alpha_7^E(M) := \alpha_8^E(N) : \Omega_7^{String}(BU(1)^{-nT}) \to \mathbb{Q}/\mathbb{Z}$$

precisely when $ch_4(E) = nx^2\, ch_2(E)$.

Theorem-in-progress (Basile–Krulewski–Leone–P.-T.)

All bordism invariants $\Omega_7^{String}(BU(1)^{-nT}) \to \mathbb{Q}/\mathbb{Z}$ arise in this way.

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Solutions: manifold generators from sphere bundles

Instead of $\mathbb{CP}^2 \times S^3$, we consider “twisted products” $\mathbb{CP}^2 \widetilde{\times} S^3$, namely sphere bundles $S(V)$ of rank 4 real vector bundles $V \to \mathbb{CP}^2$.

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- They naturally come equipped with a bulk manifold: $S(V) = \partial D(V)$.

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Theorem (Basile–Krulewski–Leone–P.-T.)

The map

$$[\mathbb{CP}^2, BSO(4)] \xrightarrow{(p_1,\chi)} \mathbb{Z} \times \mathbb{Z}$$

is an isomorphism onto the subset satisfying $\binom{p_1}{2} \equiv \chi \pmod 2$.

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- $[\mathbb{CP}^2, BU(2)] \xrightarrow[\sim]{(c_1,c_2)} \mathbb{Z} \times \mathbb{Z}$

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- Use the fibration $S^4 \to BSO(4) \to BSO(5)$ to show TS^4 acts transitively on the set of bundles with a given p_1

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Theorem (Basile–Krulewski–Leone–P.-T.)

$S(V)$ equipped with $f : S(V) \xrightarrow{\pi} \mathbb{CP}^2 \subset BU(1)$ is nc_1^2-twisted string exactly when $p_1(V) + 3 + 2n \equiv 0 \pmod{2\chi(V)}$ in which case

$$\alpha_7^E(S(V)) = \frac{ch_2(E)(p_1(V) + 3 + 2n)}{48e(V)}$$

An illustrative example: a bordism invariant for $n = 1$

A useful cofiber sequence

$$M\mathrm{String} \to M\mathrm{Spin} \to M\mathrm{Spin}/M\mathrm{String} \to \Sigma M\mathrm{String}$$

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- $M\mathrm{Spin} \wedge BU(1)^{-T}$: spin manifolds N with a map to $BU(1)$ [1]

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- $M\mathrm{Spin} \wedge BU(1)^{-T}$: spin manifolds N with a map to $BU(1)$ [1]
- $M\mathrm{Spin}/M\mathrm{String} \wedge BU(1)^{-T}$: $(N, M, f : N \to BU(1), \phi)$, N is spin, $M = \partial N$, and ϕ is a string structure on $TM - f|_{\partial N}^* T$

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Overview of the construction

We wish to construct morphisms

$$\alpha_8 : \pi_8 M\mathrm{Spin} \wedge BU(1)^{-T} \to \mathbb{Z}$$

that descend along the diagram

$$\begin{tikzcd}
\pi_8 M\mathrm{Spin} \wedge BU(1)_+ \arrow[r] \arrow[d] & \pi_8 M\mathrm{Spin}/M\mathrm{String} \wedge BU(1)^{-T} \arrow[r] \arrow[d] & \pi_7 M\mathrm{String} \wedge BU(1)^{-T} \arrow[d] \\
\mathbb{Z} \arrow[r, hook] & \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

8d integer invariant

For $(N, f) \in \pi_8 MSpin \wedge BU(1)_+$, consider the "index of the twisted Dirac operator"

$$\alpha_8^{\mathcal{O}(1)}(N, f) = \hat{A}(TN)\mathrm{ch}(f^*\mathcal{O}(1) - 1_{\mathbb{C}})[N]$$

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$$\alpha_8^{\mathcal{O}(1)}(N, f) = \hat{A}(TN)\mathrm{ch}(f^*\mathcal{O}(1) - 1_{\mathbb{C}})[N]$$

By APS, this is a K-theory pushforward[2]

$$\begin{array}{ccccc} [\mathcal{O}(1) - 1_{\mathbb{C}}] \in KU(BU(1)) & \xrightarrow{f^*} & KU(N) & \xrightarrow{i_!} & KU(\mathrm{pt}) \cong \mathbb{Z} \\ & & {\scriptstyle \hat{A}(N)\mathrm{ch}}\downarrow & & \downarrow \\ & & H(N;\mathbb{Q}) & \xrightarrow{\int_N} & H(\mathrm{pt};\mathbb{Q}) \cong \mathbb{Q} \end{array}$$

[2] Can also define as a KO pushforward

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So it defines an integer-valued map

$$\alpha_8^{\mathcal{O}(1)} : \pi_8 M\mathrm{Spin} \wedge BU(1)^{-nT} \to \mathbb{Z}$$

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Claim

$\alpha_8^{\mathcal{O}(1)}$ *can take on any integer value.*

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Proof. Consider a degree 4 hypersurface $N \subset \mathbb{CP}^5$, it naturally comes equipped with $f : N \subset \mathbb{CP}^5 \to \mathbb{CP}^\infty$.

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Proof. Consider a degree 4 hypersurface $N \subset \mathbb{CP}^5$, it naturally comes equipped with $f : N \subset \mathbb{CP}^5 \to \mathbb{CP}^\infty$. We have $TN + \mathcal{O}(1) = 6\mathcal{O}(1) - \mathcal{O}(4)$, so

$$p_1(TN) = (6 - 16)x^2 = -10x^2$$

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It follows that

$$\begin{aligned} \alpha_8^{\mathcal{O}(1)}(N, f) &= \int_N 10x^4/48 + x^4/24 = \int_N 12x^4/48 \\ &= \int_{\mathbb{P}^5} 4(12h^4/48) = 1 \end{aligned}$$

8d integer invariant

The invariant can be written as

$$\begin{aligned}\alpha_8^{\mathcal{O}(1)}(N, f) &= \hat{A}(TN)\mathrm{ch}(f^*\mathcal{O}(1) - 1_{\mathbb{C}})[N] \\ &= -\int_N (p_1(TN) - 2x^2)x^2/48 \\ &= -\int_N p_1(TN - f^*T)x^2/48\end{aligned}$$

Extending to the relative groups

Take $(N, M, f, \phi) \in \pi_8 M\mathrm{Spin}/M\mathrm{String} \wedge BU(1)^{-T}$,

$$\begin{tikzcd} & B\mathrm{String} \arrow[d] \\ M \arrow[ur, dashed, "\phi"] \arrow[r, "TM - f|_{\partial N}^* T"'] & B\mathrm{Spin} \end{tikzcd}$$

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- This defines a relative KO class $[TN - f^*T]_{\widetilde{\phi}} \in KO(N, M)$ and a relative class $p_1([TN - f^*T]_{\widetilde{\phi}}) \in H^4(N, M)$

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- Define $\alpha_{\text{rel}}^{\mathcal{O}(1)} = -p_1([TN - f^*nT]_{\widetilde{\phi}})c_1(\mathcal{L})^2[N]$ using Poincare-Lefschetz duality

Extending to the relative groups

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Descending to a 7d invariant

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- Set $\alpha_7^{\mathcal{O}(1)}(M, f, \phi) := \alpha_{\mathrm{rel}}^{\mathcal{O}(1)}(N, M)$

Summary

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Index of Dirac operator twisted by $\mathcal{O}(1) - 1_{\mathbb{C}}$ is a K-theory pushforward.

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It factors a $p_1([TN - f^*T])$ so it extends to a relative invariant.

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Any twisted string 7-manifold is on the boundary of a spin 8-manifold so this descends to a 7d invariant.

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General case: need to choose E to twist Dirac operator by that both descends and detects the most torsion.

Manifold generators

Twisted string structures on $S(V)$

Given a rank 4 bundle $V_{p,\chi} \xrightarrow{p} \mathbb{CP}^2$ with $p = p_1(V_{p,\chi})$, $\chi = e(V_{p,\chi})$, $\binom{p}{2} \equiv \chi \pmod 2$

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$$KO(D(V), S(V)) \to \widetilde{KO}(D(V)) \ni [TD(V) - f^*T]$$

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$$KO(D(V), S(V)) \to \widetilde{KO}(D(V)) \ni [TD(V) - f^* T]$$

$$[TD(V) - f^* T] = [V + 3\mathcal{O}(1) - (\mathcal{O}(1) + \mathcal{O}(-1))]$$

$\alpha_7^{\mathcal{O}(1)}$ on $-T$-twisted sphere bundles

Theorem (Basile–Krulewski–Leone–P.-T.)

Given a rank 4 bundle $V_{p,\chi} \xrightarrow{p} \mathbb{CP}^2$ with $p = p_1(V_{p,\chi}), \chi = e(V_{p,\chi})$, $S(V_{p,\chi})$ admits a $(BU(1), -T)$-twisted string structure exactly when

$$p + 1 \equiv 0 \pmod{2\chi}$$

in which case

$$\alpha_7(S(V_{p,\chi})) = -\frac{p+1}{48\chi}$$

α_7^E on $-nT$-twisted sphere bundles

Theorem (Basile–Krulewski–Leone–P.-T.)

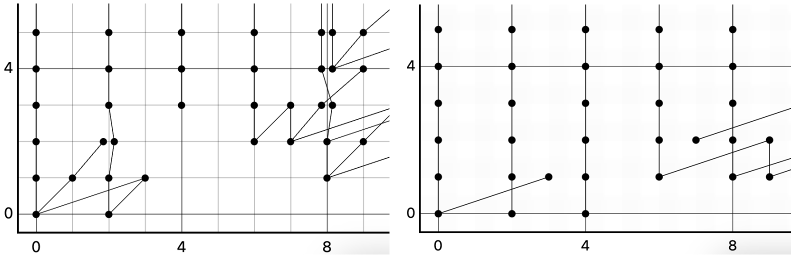
Given a rank 4 bundle $V_{p,\chi} \xrightarrow{p} \mathbb{CP}^2$ *with* $p = p_1(V_{p,\chi}), \chi = e(V_{p,\chi})$, $S(V_{p,\chi})$ *admits a* $(BU(1), -T)$*-twisted string structure exactly when*

$$p + 3 - 2n \equiv 0 \pmod{2\chi}$$

in which case

$$\alpha_7^E = -\frac{ch_2(E)(p + 3 - 2n)}{48\chi}$$

Order of the $n = 1$ bordism group



- $p = 3, \chi = 1$ satisfies $\binom{3}{2} \equiv 1 \pmod 2$, $3 + 1 \equiv 0 \mod 2$ and

$$\alpha_7^{\mathcal{O}(1)}(S(V_{3,1})) = -\frac{1}{12}$$

Thank you!